

On balanced incomplete block designs with specified weak chromatic number

Daniel Horsley
 School of Mathematical Sciences
 Monash University
 Vic 3800, Australia
 danhorsley@gmail.com

David A. Pike
 Department of Mathematics and Statistics
 Memorial University of Newfoundland
 St. John's, NL, Canada A1C 5S7
 dapike@mun.ca

Abstract

A weak c -colouring of a balanced incomplete block design (BIBD) is a colouring of the points of the design with c colours in such a way that no block of the design has all of its vertices receive the same colour. A BIBD is said to be weakly c -chromatic if c is the smallest number of colours with which the design can be weakly coloured. In this paper we show that for all $c \geq 2$ and $k \geq 3$ with $(c, k) \neq (2, 3)$, the obvious necessary conditions for the existence of a (v, k, λ) -BIBD are asymptotically sufficient for the existence of a weakly c -chromatic (v, k, λ) -BIBD.

1 Introduction

A *balanced incomplete block design* of order v , block size k and index λ , denoted a (v, k, λ) -BIBD, is a pair (V, \mathcal{B}) such that V is a set of v elements (called points) and \mathcal{B} is a collection of k element subsets of V (called blocks) such that each unordered pair of points in V is contained in exactly λ blocks in \mathcal{B} . A *partial* (v, k, λ) -BIBD is defined similarly except that each pair of points in V must be contained in at most λ blocks in \mathcal{B} .

For a positive integer c , a weak c -colouring of a (partial) (v, k, λ) -BIBD is a colouring of the points of the design with c colours in such a way that no block of the design has all of its points receive the same colour. A (partial) (v, k, λ) -BIBD is said to be *weakly c -chromatic*, or to have *weak chromatic number c* , if c is the smallest number of colours with which the design can be weakly coloured. Since weak colourings are the only colourings of designs we will consider in this paper, we will often omit the adjectives ‘weak’ and ‘weakly’ in what follows.

It is obvious that if there exists a (v, k, λ) -BIBD then

- (i) $\lambda(v - 1) \equiv 0 \pmod{k - 1}$; and
- (ii) $\lambda v(v - 1) \equiv 0 \pmod{k(k - 1)}$.

Wilson [20] famously proved that (i) and (ii) are asymptotically sufficient for the existence of a (v, k, λ) -BIBD. That is, for any positive integers k and λ with $k \geq 3$, there exists an integer $N'(k, \lambda)$ such that if $v \geq N'(k, \lambda)$ then (i) and (ii) are sufficient for the existence of a (v, k, λ) -BIBD. In this paper, we will extend Wilson’s result to c -chromatic BIBDs by showing that, for any positive integers c , k and λ , such that $c \geq 2$, $k \geq 3$ and $(k, c) \neq (3, 2)$, (i) and (ii) are asymptotically sufficient for the existence of a c -chromatic (v, k, λ) -BIBD. For the sake of brevity, we will call positive integers v which satisfy (i) and (ii) (k, λ) -admissible. Note that if an integer v is (k, λ) -admissible then so is every positive integer congruent to v modulo $k(k - 1)$.

Weak colourings were first introduced in the context of hypergraphs, and this naturally led to the study of weak colourings of block designs. A simple counting argument [16] shows that 2-chromatic $(v, 3, \lambda)$ -BIBDs exist only for $v \leq 4$. It has been shown by de Brandes, Phelps and Rödl [4] that for all integers $c \geq 3$ there is an integer $N(c, 3, 1)$ such that for all $(3, 1)$ -admissible integers $v \geq N(c, 3, 1)$ there is a c -chromatic $(v, 3, 1)$ -BIBD. For a positive integer λ it is known that a 2-chromatic $(v, 4, \lambda)$ -BIBD exists for each $(4, \lambda)$ -admissible integer v , with almost all of the problem solved in [10] and [11] and the outstanding cases resolved in [17] and [7]. Furthermore, Ling [14], has shown that a 2-chromatic $(v, 5, 1)$ -BIBD exists for each $(5, 1)$ -admissible integer v . For a survey of colourings of block designs see [17]. The main result of this paper is as follows.

Theorem 1.1 *Let c , k and λ be positive integers such that $c \geq 2$, $k \geq 3$ and $(c, k) \neq (2, 3)$. Then there is an integer $N(c, k, \lambda)$ such that there exists a weakly c -chromatic (v, k, λ) -BIBD for all (k, λ) -admissible integers $v \geq N(c, k, \lambda)$.*

In Section 2 we give some definitions that we will require throughout the paper and prove a number of preliminary results. Sections 3, 4 and 5 deal with BIBDs with block size at least 4. In Section 3 we find various examples of 2-chromatic BIBDs, and these are then used in Section 4 to obtain various examples of c -chromatic BIBDs for each $c \geq 2$. In Section 5 we are then able to use results from Sections 2, 3 and 4 to demonstrate the asymptotic existence of c -chromatic BIBDs for each $c \geq 2$. Finally, in Section 6, we deal with the case of BIBDs with block size 3 (that is, triple systems) and thus complete the proof of Theorem 1.1.

2 Preliminary definitions and results

Let v and λ be positive integers and let K be a set of positive integers. A *group divisible design* of order v and index λ with block sizes from K , denoted a (K, λ) -GDD, is a triple $(V, \mathcal{G}, \mathcal{B})$ such that V is a set of v elements (called points), \mathcal{G} is a partition of V into parts (called groups) and \mathcal{B} is a collection of subsets of V (called blocks) such that $|B| \in K$ for all $B \in \mathcal{B}$, each unordered pair of points in different groups is contained in exactly λ blocks, and no unordered pair of points in the same group is contained in any block. If, for integers g_1, g_2, \dots, g_t and a_1, a_2, \dots, a_t , \mathcal{G} contains a_i groups of size g_i for $i \in \{1, 2, \dots, t\}$ and \mathcal{G} contains no groups of any other size then we say that $(V, \mathcal{G}, \mathcal{B})$ is of type $g_1^{a_1} g_2^{a_2} \dots g_t^{a_t}$. We will abbreviate $(\{k\}, \lambda)$ -GDD to (k, λ) -GDD. A $(k, 1)$ -GDD of type g^k is more commonly referred to as a transversal design with group size g and block size k .

We say that a partial BIBD (V_1, \mathcal{B}_1) is *embedded* in a partial BIBD (V_2, \mathcal{B}_2) if $V_1 \subseteq V_2$ and $\mathcal{B}_1 \subseteq \mathcal{B}_2$. A *decomposition* of a graph G is a collection $\{G_1, G_2, \dots, G_t\}$ of subgraphs of G whose edge sets partition the edge set of G . We extend this definition to edge-coloured graphs in the obvious way. A (v, k, λ) -BIBD can be considered as a decomposition of the λ -fold complete graph with v vertices into copies of the complete graph with k vertices.

To simplify the presentation of many of our results, we will introduce a generalisation of the well-known concept of a blocking set. We will say that a collection $\{S_1, S_2, \dots, S_c\}$ of pairwise disjoint subsets of the point set of a design is a *blocking system* for that design if each block of the design has a non-empty intersection with at least two of the sets in $\{S_1, S_2, \dots, S_c\}$. We will also refer to such a blocking system as a c -blocking system if we wish to specify the number of sets in the system or as an (s_1, s_2, \dots, s_c) -blocking system, where $s_i = |S_i|$ for each $i \in \{1, 2, \dots, c\}$, if we wish to specify the sizes of the sets in the system. Obviously the existence of a c -blocking

system for a design implies the existence of a c -colouring for that design. Note that if a design on v points has an (s_1, s_2, \dots, s_c) -blocking system for integers s_1, s_2, \dots, s_c then it has an $(s'_1, s'_2, \dots, s'_c)$ -blocking system for all integers s'_1, s'_2, \dots, s'_c such that $s'_1 + s'_2 + \dots + s'_c \leq v$ and $s'_i \geq s_i$ for each $i \in \{1, 2, \dots, c\}$. We will often use this fact tacitly in what follows.

Our main goal in this section will be to prove Lemma 2.2. Our proof uses techniques from [13] and closely follows the proof of Theorem 8.1 of that paper, although we must be careful at times to ensure that the GDD we obtain has the required blocking system. We will require the following well-known result (see [18], for example).

Lemma 2.1 *Let r be a positive integer. Given a set of r -dimensional rational vectors U , an r -dimensional rational vector \mathbf{c} can be written as an integral combination of the vectors in U if and only if, for every r -dimensional rational vector \mathbf{y} such that the dot product $\mathbf{y} \cdot \mathbf{u}$ is an integer for each $\mathbf{u} \in U$, the dot product $\mathbf{y} \cdot \mathbf{c}$ is an integer.*

Lemma 2.2 *Let k, λ and g be positive integers such that $k \geq 4$, either $g = k - 1$ or $g \geq 2k - 2$, and if $k = 4$ then g is even. Then for each sufficiently large integer t satisfying*

$$(i) \quad \lambda g(t - 1) \equiv 0 \pmod{k - 1} \text{ and}$$

$$(ii) \quad \lambda g^2 t(t - 1) \equiv 0 \pmod{k(k - 1)},$$

there exists a (k, λ) -GDD of type g^t which has a 2-blocking system such that each set of the blocking system intersects each group of the GDD in exactly $\lfloor \frac{g}{2} \rfloor$ points.

Proof We first introduce some notation that we will use throughout this proof. We denote by $\lambda K_n^{(R)}$ the edge-coloured digraph on n vertices such that for each ordered pair of vertices (x, y) and each colour $r \in R$ there are exactly λ directed edges of colour r from x to y . We will denote by $\mathbf{1}_n$ the n -dimensional vector all of whose components are 1. Furthermore, we will adopt the convention that if \mathbf{u} is an n -dimensional vector then, unless we specify otherwise, \mathbf{u} is indexed by $\{1, 2, \dots, n\}$ and component i of \mathbf{u} is represented by u_i . For rational numbers x and y we shall use the notation $x \equiv y$ to indicate that $x - y$ is an integer.

Let $G = \{1, 2, \dots, g\}$, let $G_1 = \{1, 2, \dots, \lfloor \frac{g}{2} \rfloor\}$, let $G_2 = \{\lfloor \frac{g}{2} \rfloor + 1, \lfloor \frac{g}{2} \rfloor + 2, \dots, 2\lfloor \frac{g}{2} \rfloor\}$. Let $R = G \times G$ be a set of colours. Let F be the set of all g -dimensional integral vectors \mathbf{f} such that

- $f_i \geq 0$ for each $i \in \{1, 2, \dots, g\}$;

- $f_1 + f_2 + \cdots + f_g = k$; and
- $f_1 + f_2 + \cdots + f_{\lfloor \frac{g}{2} \rfloor} \geq 1$ and $f_{\lfloor \frac{g}{2} \rfloor + 1} + f_{\lfloor \frac{g}{2} \rfloor + 2} + \cdots + f_{2\lfloor \frac{g}{2} \rfloor} \geq 1$.

For each vector $\mathbf{f} \in F$ let $H_{\mathbf{f}}$ be the edge-coloured digraph with k vertices such that

- $V(H_{\mathbf{f}})$ has an ordered partition (V_1, V_2, \dots, V_g) such that $|V_i| = f_i$ for each $i \in \{1, 2, \dots, g\}$; and
- for any ordered pair of distinct vertices (x, y) from $V(H_{\mathbf{f}})$ there is exactly one directed edge from x to y and it has colour (i, j) , where i and j are the unique elements of G such that $x \in V_i$ and $y \in V_j$.

Let $\mathcal{H} = \{H_{\mathbf{f}} : \mathbf{f} \in F\}$.

It can be seen that an (edge-coloured) decomposition of $\lambda K_t^{(R)}$ into copies of graphs in \mathcal{H} will yield a (k, λ) -GDD of type g^t (see [13] for details) and furthermore that this (k, λ) -GDD will have a 2-blocking system such that each set of the blocking system intersects each group of the GDD in exactly $\lfloor \frac{g}{2} \rfloor$ points. So it suffices to show that for each sufficiently large integer t satisfying (i) and (ii), there is a decomposition of $\lambda K_t^{(R)}$ into copies of graphs in \mathcal{H} .

For an edge-coloured digraph H , let $\mu(H)$ be a (g^2) -dimensional vector, indexed by R , whose (i, j) component is the number of directed edges in H of colour (i, j) . For an edge-coloured digraph H and a vertex x of H , let $\tau(H, x)$ be a $(2g^2)$ -dimensional vector, indexed by $R \times \{1, 2\}$, whose $((i, j), 1)$ component is the number of directed edges to x of colour (i, j) and whose $((i, j), 2)$ component is the number of directed edges from x of colour (i, j) .

Then by Theorem 13.1 of [13] it suffices to prove that, for each sufficiently large integer t satisfying (i) and (ii),

- (a) $\lambda t(t-1)\mathbf{1}_{g^2}$ is an integral linear combination of vectors in $\{\mu(H) : H \in \mathcal{H}\}$;
- (b) $\lambda(t-1)\mathbf{1}_{2g^2}$ is an integral linear combination of vectors in $\{\tau(H, y) : H \in \mathcal{H} \text{ and } y \in V(H)\}$; and
- (c) $\mathbf{1}_{g^2}$ is a non-negative rational linear combination of vectors in $\{\mu(H) : H \in \mathcal{H}\}$.

Let t be a positive integer satisfying (i) and (ii). We will prove (a), (b) and (c) separately.

Proof of (a). For each $\mathbf{f} \in F$, the (i, i) component of $\mu(H_{\mathbf{f}})$ is $f_i(f_i - 1)$ for all $i \in \{1, 2, \dots, g\}$ and the (i, j) component of $\mu(H_{\mathbf{f}})$ is $f_i f_j$ for all $i, j \in \{1, 2, \dots, g\}$ such that $i \neq j$. Thus by Lemma 2.1 it suffices to prove that for any list of g^2 rational numbers $\{x_{ij}\}_{i,j \in \{1, 2, \dots, g\}}$ satisfying

$$\sum_{i \neq j} f_i f_j x_{ij} + \sum_i f_i(f_i - 1)x_{ii} \equiv 0 \quad \text{for each } \mathbf{f} \in F, \quad (1)$$

we have that

$$\sum_{i,j} \lambda t(t-1)x_{ij} \equiv 0.$$

Let a and b be distinct elements of G . Let c be an element of $G \setminus \{a, b\}$ such that if $k \geq 5$ then $\{a, b, c\} \cap G_1 \neq \emptyset$ and $\{a, b, c\} \cap G_2 \neq \emptyset$, and if $k = 4$ then $\{b, c\} \cap G_1 \neq \emptyset$ and $\{b, c\} \cap G_2 \neq \emptyset$ (note that g is even if $k = 4$). Let \mathbf{f}' be the vector in F such that $f'_a = k - 2$, and $f'_b = f'_c = 1$. Let \mathbf{f}'' be the vector in F such that $f''_a = k - 3$, $f''_b = 2$ and $f''_c = 1$. Let \mathbf{f}''' be the vector in F such that $f'''_a = k - 4$, $f'''_b = 3$ and $f'''_c = 1$. Subtracting twice the congruence implied by (1) when $\mathbf{f} = \mathbf{f}''$ from the sum of the two congruences implied by (1) when $\mathbf{f} = \mathbf{f}'$ and $\mathbf{f} = \mathbf{f}'''$ we see that

$$2x_{ab} + 2x_{ba} \equiv 2x_{aa} + 2x_{bb}.$$

Thus,

$$2x_{ij} + 2x_{ji} \equiv 2x_{ii} + 2x_{jj} \quad \text{for all } i, j \in G, \quad (2)$$

noting that the congruence is true trivially if $i = j$.

Let $a \in G \setminus \{1\}$ and let $b \in G_2 \setminus \{a\}$. If k is odd, let \mathbf{f}^\dagger be the vector in F such that $f_1^\dagger = \frac{k-1}{2}$, $f_a^\dagger = \frac{k-3}{2}$ and $f_b^\dagger = 2$, and let \mathbf{f}^\ddagger be the vector in F such that $f_1^\ddagger = \frac{k-3}{2}$, $f_a^\ddagger = \frac{k-1}{2}$ and $f_b^\ddagger = 2$. If k is even, let \mathbf{f}^\dagger be the vector in F such that $f_1^\dagger = \frac{k}{2}$, $f_a^\dagger = \frac{k-2}{2}$ and $f_b^\dagger = 1$, and let \mathbf{f}^\ddagger be the vector in F such that $f_1^\ddagger = \frac{k-2}{2}$, $f_a^\ddagger = \frac{k}{2}$ and $f_b^\ddagger = 1$. Subtracting the congruence implied by (1) when $\mathbf{f} = \mathbf{f}^\ddagger$ from the congruence implied by (1) when $\mathbf{f} = \mathbf{f}^\dagger$, doubling the resulting congruence if k is even, and then using (2) we see that

$$\begin{aligned} (k-1)x_{11} &\equiv (k-1)x_{aa} \quad \text{if } k \text{ is odd, and} \\ 2(k-1)x_{11} &\equiv 2(k-1)x_{aa} \quad \text{if } k \text{ is even.} \end{aligned}$$

Thus,

$$\begin{aligned} (k-1)x_{11} &\equiv (k-1)x_{ii} \quad \text{for all } i \in G \text{ if } k \text{ is odd, and} \\ 2(k-1)x_{11} &\equiv 2(k-1)x_{ii} \quad \text{for all } i \in G \text{ if } k \text{ is even.} \end{aligned} \quad (3)$$

Let $a \in G_2$ and let \mathbf{f}^* be the vector in F such that $f_1^* = k - 2$ and $f_a^* = 2$. Using both (2) and (3), it is easy to see from the congruence implied by (1) when $\mathbf{f} = \mathbf{f}^*$ that $k(k - 1)x_{11} \equiv 0$ and thus, since t satisfies (ii), we have

$$\lambda g^2 t(t - 1)x_{11} \equiv 0. \quad (4)$$

So, using (2), (3) and (4), noting that $\lambda t(t - 1)$ is a multiple of 2, that $\lambda g t(t - 1)$ is a multiple of $k - 1$ if k is odd (by (i)), and that $\lambda g t(t - 1)$ is a multiple of $2(k - 1)$ if k is even (by (i)), we have

$$\sum_{i,j} \lambda t(t - 1)x_{ij} \equiv \sum_i \lambda g t(t - 1)x_{ii} \equiv \lambda g^2 t(t - 1)x_{11} \equiv 0$$

as required.

Proof of (b). Let \mathbf{f} be a vector in F , let x be a vertex of $H_{\mathbf{f}}$ and let ℓ be the element of G such that $x \in V_{\ell}$ (where (V_1, V_2, \dots, V_g) is the ordered partition of $V(H_{\mathbf{f}})$ in the definition of $H_{\mathbf{f}}$). Then the $((\ell, \ell), 1)$ and $((\ell, \ell), 2)$ components of $\tau(H_{\mathbf{f}}, x)$ are $f_{\ell} - 1$, the $((i, \ell), 1)$ and $((\ell, i), 2)$ components of $\tau(H_{\mathbf{f}}, x)$ are f_i for all $i \in G \setminus \{\ell\}$, and all the other components of $\tau(H_{\mathbf{f}}, x)$ are 0. Thus by Lemma 2.1 it suffices to prove that for any list of $2g^2$ rational numbers $\{x_{ij}, y_{ij}\}_{i,j \in \{1, 2, \dots, g\}}$ satisfying

$$(f_{\ell} - 1)(x_{\ell\ell} + y_{\ell\ell}) + \sum_{i \neq \ell} f_i(x_{i\ell} + y_{\ell i}) \equiv 0 \quad \text{for each } \mathbf{f} \in F \text{ and } \ell \in G \text{ such that } f_{\ell} \geq 1, \quad (5)$$

we have that

$$\sum_{i,j} \lambda(t - 1)(x_{ij} + y_{ij}) \equiv 0.$$

Let a and b be distinct elements of G . Let c be an element of $G \setminus \{a, b\}$ such that $\{a, b, c\} \cap G_1 \neq \emptyset$ and $\{a, b, c\} \cap G_2 \neq \emptyset$. Let \mathbf{f}' be the vector in F such that $f'_a = k - 2$, and $f'_b = f'_c = 1$. Let \mathbf{f}'' be the vector in F such that $f''_a = k - 3$, $f''_b = 2$ and $f''_c = 1$. Subtracting the congruence implied by (5) when $\mathbf{f} = \mathbf{f}''$ and $\ell = a$ from the congruence implied by (5) when $\mathbf{f} = \mathbf{f}'$ and $\ell = a$ we see that

$$x_{aa} + y_{aa} \equiv x_{ba} + y_{ab}.$$

Thus,

$$x_{ii} + y_{ii} \equiv x_{ji} + y_{ij} \quad \text{for all } i, j \in G. \quad (6)$$

Using (6), it is easy to see from the congruence implied by (5) when $\mathbf{f} = \mathbf{f}'$ and $\ell = a$ that $(k - 1)(x_{aa} + y_{aa}) \equiv 0$. Thus, we have

$$(k - 1)(x_{ii} + y_{ii}) \equiv 0 \quad \text{for all } i \in G. \quad (7)$$

So, using (6) and (7), noting that $\lambda g(t-1)$ is a multiple of $k-1$ (by (i)), we have

$$\sum_{i,j} \lambda(t-1)(x_{ij} + y_{ij}) \equiv \sum_i \lambda g(t-1)(x_{ii} + y_{ii}) \equiv 0$$

as required.

Proof of (c). We first suppose $g = k-1$. The case where $g \geq 2k-2$ will be dealt with later. Note that in this case $k \geq 5$. Let F' be the set of all g -dimensional vectors with exactly one component equal to $\lceil \frac{k}{2} \rceil$, exactly $\lfloor \frac{k}{2} \rfloor$ components equal to 1, and all other components equal to 0. Note that $F' \subseteq F$. Through routine counting, it can be calculated that the (i, j) component of the vector $\sum_{\mathbf{f} \in F'} \mu(H_{\mathbf{f}})$ is $|F'|x_1$ if $i \neq j$ and is $|F'|x_2$ if $i = j$, where

$$x_1 = \frac{1}{(k-1)(k-2)}(2\lceil \frac{k}{2} \rceil \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{2} \rfloor \lfloor \frac{k-2}{2} \rfloor) \quad \text{and} \quad x_2 = \frac{1}{k-1} \lceil \frac{k}{2} \rceil \lceil \frac{k-2}{2} \rceil.$$

Let F'' be the set of all g -dimensional vectors with exactly one component equal to 2 and every other component equal to 1. Note that $F'' \subseteq F$. It can be calculated that the (i, j) component of the vector $\sum_{\mathbf{f} \in F''} \mu(H_{\mathbf{f}})$ is $|F''|y_1$ if $i \neq j$ and is $|F''|y_2$ if $i = j$, where

$$y_1 = \frac{k+1}{k-1} \quad \text{and} \quad y_2 = \frac{2}{k-1}.$$

It is routine to check that $y_1 \geq y_2$ and $x_1 \leq x_2$ for all integers $k \geq 5$, and thus it is easy to see that $\mathbf{1}_{g^2}$ is a non-negative rational linear combination of vectors in $\{\mu(H_{\mathbf{f}}) : \mathbf{f} \in F\}$, as required.

Now we suppose that g is odd and $g \geq 2k-1$. The case where g is even and $g \geq 2k-2$ will be dealt with later. Let ℓ be the integer such that $g = 2\ell + 1$ and note that $\ell \geq k-1$. Also note that, since g is odd, $k \geq 5$. We define F_1, F_2, F_3, F_4 and F_5 to be subsets of F , as follows.

$$\begin{aligned} F_1 &= \{\mathbf{f} \in F : f_i = k-1 \text{ for some } i \in G_1 \cup G_2\} \\ F_2 &= \{\mathbf{f} \in F : f_i \leq 1 \text{ for all } i \in G, \{\sum_{i \in G_1} f_i, \sum_{i \in G_2} f_i\} = \{1, k-1\}\} \\ F_3 &= \{\mathbf{f} \in F : f_i \leq 1 \text{ for all } i \in G, \{\sum_{i \in G_1} f_i, \sum_{i \in G_2} f_i\} = \{\lfloor \frac{k}{2} \rfloor, \lceil \frac{k}{2} \rceil\}\} \\ F_4 &= \{\mathbf{f} \in F : f_g = k-2\} \\ F_5 &= \{\mathbf{f} \in F : f_i \leq 1 \text{ for all } i \in G, \{\sum_{i \in G_1} f_i, \sum_{i \in G_2} f_i\} = \{\lfloor \frac{k-1}{2} \rfloor, \lceil \frac{k-1}{2} \rceil\}\} \end{aligned}$$

In the remainder of this case, we will say that a vector indexed by $G \times G$ is of type $(z_1, z_2, z_3, z_4, z_5)$

if for all $(i, j) \in G \times G$ its (i, j) component is u_{ij} where

$$u_{ij} = \begin{cases} z_1, & \text{if } i = j \text{ and } i \in G_1 \cup G_2; \\ z_2, & \text{if } i \neq j \text{ and either } \{i, j\} \subseteq G_1 \text{ or } \{i, j\} \subseteq G_2; \\ z_3, & \text{if } i \neq j \text{ and either } (i, j) \in G_1 \times G_2 \text{ or } (j, i) \in G_1 \times G_2; \\ z_4, & \text{if } i = j = g; \\ z_5, & \text{if } i \neq j \text{ and either } i = g \text{ or } j = g. \end{cases}$$

Through routine but tedious counting it can be calculated that $\sum_{\mathbf{f} \in F_i} \mu(H_{\mathbf{f}})$ is of type $|F_1|\mathbf{a}$ if $i = 1$, $|F_2|\mathbf{b}$ if $i = 2$, $|F_3|\mathbf{c}$ if $i = 3$, $|F_4|\mathbf{d}$ if $i = 4$, and $|F_5|\mathbf{e}$ if $i = 5$, where

$$\begin{aligned} \mathbf{a} &= \left(\frac{(k-1)(k-2)}{2\ell}, 0, \frac{k-1}{\ell^2}, 0, 0 \right), \\ \mathbf{b} &= \left(0, \frac{(k-1)(k-2)}{2\ell(\ell-1)}, \frac{k-1}{\ell^2}, 0, 0 \right), \\ \mathbf{c} &= \left(0, \frac{1}{2\ell(\ell-1)}(\lceil \frac{k}{2} \rceil \lceil \frac{k-2}{2} \rceil + \lfloor \frac{k}{2} \rfloor \lfloor \frac{k-2}{2} \rfloor), \frac{1}{\ell^2}(\lceil \frac{k}{2} \rceil \lfloor \frac{k}{2} \rfloor), 0, 0 \right), \\ \mathbf{d} &= \left(0, 0, \frac{1}{\ell^2}, (k-2)(k-3), \frac{k-2}{\ell} \right), \\ \mathbf{e} &= \left(0, \frac{1}{2\ell(\ell-1)}(\lceil \frac{k-1}{2} \rceil \lceil \frac{k-3}{2} \rceil + \lfloor \frac{k-1}{2} \rfloor \lfloor \frac{k-3}{2} \rfloor), \frac{1}{\ell^2}(\lceil \frac{k-1}{2} \rceil \lfloor \frac{k-1}{2} \rfloor), 0, \frac{k-1}{2\ell} \right). \end{aligned}$$

Obviously then to show that $\mathbf{1}_{g^2}$ is a non-negative rational linear combination of vectors in $\{\mu(H) : H \in \mathcal{H}\}$ it suffices to show that $(1, 1, 1, 1, 1)$ is a non-negative rational combination of \mathbf{a} , \mathbf{b} , \mathbf{c} , \mathbf{d} and \mathbf{e} .

Simple calculations give us that

$$\frac{2\ell}{(k-1)(k-2)}\mathbf{a} + \frac{1}{(k-2)(k-3)}\mathbf{d} + \frac{2\ell(k-3)-2}{(k-1)(k-3)}\mathbf{e} = (1, x_2, x_3, 1, 1) \quad \text{where}$$

$$\begin{aligned} x_2 &= \frac{\ell(k-3)-1}{\ell(\ell-1)(k-1)(k-3)}(\lceil \frac{k-1}{2} \rceil \lceil \frac{k-3}{2} \rceil + \lfloor \frac{k-1}{2} \rfloor \lfloor \frac{k-3}{2} \rfloor) \quad \text{and} \\ x_3 &= \frac{1}{\ell^2(k-2)(k-3)} + \frac{2}{\ell(k-2)} + \frac{2(\ell(k-3)-1)}{\ell^2(k-1)(k-3)}(\lceil \frac{k-1}{2} \rceil \lfloor \frac{k-1}{2} \rfloor). \end{aligned}$$

So it suffices to show that $(0, 1 - x_2, 1 - x_3, 0, 0)$ is a non-negative rational combination of \mathbf{b} and \mathbf{c} and hence, since b_2 , $(1 - x_2)$ and c_2 are all positive, it suffices to show that

$$\frac{b_3}{b_2} \leq \frac{1 - x_3}{1 - x_2} \leq \frac{c_3}{c_2}.$$

Let $\Delta_1 = b_2(1 - x_3) - b_3(1 - x_2)$ and $\Delta_2 = c_3(1 - x_2) - c_2(1 - x_3)$. We will show that Δ_1 and Δ_2 are both non-negative. Substituting in for b_2 , b_3 , c_2 , c_3 , x_2 and x_3 and simplifying yields that

when k is even

$$\Delta_1 = \frac{1}{4\ell^3(\ell-1)(k-3)}(2\ell(\ell-k+1)(k-1)(k-3)(k-4) + k(\ell(k-3)+1)(k^2-6k+6) + (4k-6)) \text{ and}$$

$$\Delta_2 = \frac{k}{4\ell^3(\ell-1)(k-3)}(2\ell(\ell-k+1)(k-3) + k\ell(k-3)+1),$$

and when k is odd

$$\Delta_1 = \frac{k-1}{4\ell^3(\ell-1)}(2\ell(\ell-k+1)(k-4) + k\ell(k-5) + k-2) \text{ and}$$

$$\Delta_2 = \frac{k-1}{4\ell^3(\ell-1)(k-2)}(2\ell(\ell-k+1)(k-2) + k\ell(k-1) - 1).$$

Given that $k \geq 5$ and that $\ell \geq k-1$, it is now routine to confirm that Δ_1 and Δ_2 are non-negative, as required.

If $g \geq k+1$ and g is even then a similar (but slightly easier) argument can be made to show that $\mathbf{1}_{g^2}$ is a non-negative rational linear combination of vectors in $\{\mu(H_{\mathbf{f}}) : \mathbf{f} \in F_1 \cup F_2 \cup F_3\}$ (where F_1 , F_2 and F_3 are defined as above). \square

Note that in the above lemma the condition that if $k=4$ then g is even is necessary. This can be seen by observing that if $\{S_1, S_2\}$ is a blocking system for a design with block size 4 then every block of the design contains at least three pairs of points which intersect S_1 in exactly one point, and thus at least half of the pairs of points which appear in blocks of the design must intersect S_1 in exactly one point. On the other hand, with more work, the restriction that either $g = k-1$ or $g \geq 2k-2$ could certainly be loosened. The above result suffices for our purposes here, however.

By combining Lemma 2.2 with some standard “group filling” constructions, we can obtain the following two results.

Lemma 2.3 *Let y , k and λ be positive integers such that $k \geq 4$, either $y = k$ or $y \geq 2k-1$, and if $k=4$ then y is odd. If there exists a (y, k, λ) -BIBD which has a $(\lfloor \frac{y-1}{2} \rfloor, \lfloor \frac{y-1}{2} \rfloor)$ -blocking system, then, for each sufficiently large integer x such that $x(y-1)+1$ is (k, λ) -admissible,*

- (a) *there exists an $(x(y-1)+1, k, \lambda)$ -BIBD which has an $(x\lfloor \frac{y-1}{2} \rfloor, x\lfloor \frac{y-1}{2} \rfloor)$ -blocking system; and*
- (b) *there exists a (k, λ) -GDD of type $y^1 1^{(x-1)(y-1)}$ which has an $(x\lfloor \frac{y-1}{2} \rfloor, x\lfloor \frac{y-1}{2} \rfloor)$ -blocking system such that each set of the blocking system intersects the group of size y in exactly $\lfloor \frac{y-1}{2} \rfloor$ points.*

Proof Since y is (k, λ) -admissible, it is easy to check that, for a sufficiently large integer x such that $x(y-1)+1$ is (k, λ) -admissible, Lemma 2.2 implies that there exists a (k, λ) -GDD $(V, \mathcal{G}, \mathcal{A})$ of type $(y-1)^x$ which has a 2-blocking system $\{S_1, S_2\}$ such that $|S_1 \cap G| = |S_2 \cap G| = \lfloor \frac{y-1}{2} \rfloor$ for each $G \in \mathcal{G}$. Now let ∞ be a point not in V , let $G^* \in \mathcal{G}$ and for each $G \in \mathcal{G}$ let \mathcal{A}_G be a collection of blocks such that $(G \cup \{\infty\}, \mathcal{A}_G)$ is a (y, k, λ) -BIBD for which $\{S_1 \cap G, S_2 \cap G\}$ is a blocking system. Let

$$\mathcal{B} = \mathcal{A} \cup \bigcup_{G \in \mathcal{G}} \mathcal{A}_G.$$

Then $(V \cup \{\infty\}, \mathcal{B})$ is the required BIBD, $(V \cup \{\infty\}, \{G^* \cup \{\infty\}\} \cup \{\{z\} : z \in V \setminus G^*\}, \mathcal{B} \setminus \mathcal{A}_{G^*})$ is the required GDD, and in both cases $\{S_1, S_2\}$ is the required blocking system. \square

Lemma 2.4 *Let y, k and λ be positive integers such that $k \geq 4$, $y \geq 2k-2$ and if $k=4$ then y is even. If there exists a (y, k, λ) -BIBD which has a $(\lfloor \frac{y}{2} \rfloor, \lfloor \frac{y}{2} \rfloor)$ -blocking system then, for each sufficiently large integer x such that xy is (k, λ) -admissible,*

- (a) *there exists an (xy, k, λ) -BIBD which has an $(x\lfloor \frac{y}{2} \rfloor, x\lfloor \frac{y}{2} \rfloor)$ -blocking system; and*
- (b) *there exists a (k, λ) -GDD of type $y^1 1^{(x-1)y}$ which has an $(x\lfloor \frac{y}{2} \rfloor, x\lfloor \frac{y}{2} \rfloor)$ -blocking system such that each set of the blocking system intersects the group of size y in exactly $\lfloor \frac{y}{2} \rfloor$ points.*

Proof This is proved very similarly to Lemma 2.3, except that we take a base GDD of type y^x and we do not add the point ∞ . \square

3 Examples of 2-chromatic BIBDs

In this section we will use Lemmas 2.3 and 2.4 to find various examples of 2-chromatic BIBDs. In Lemma 3.1 we establish, for all positive integers k and λ with $k \geq 5$, the asymptotic existence of 2-chromatic BIBDs with block size k , index λ and order congruent to 1 modulo $k-1$. We then construct, for all positive integers k and λ with $k \geq 4$, 2-chromatic BIBDs with block size k and index λ whose orders fall in each admissible congruence class modulo $k(k-1)$. This is accomplished in Lemma 3.2 for $k \geq 5$ and in Lemma 3.3 for $k=4$.

Lemma 3.1 *Let k and λ be positive integers such that $k \geq 5$. For each sufficiently large (k, λ) -admissible integer v such that $v \equiv 1 \pmod{k-1}$, there exists a (v, k, λ) -BIBD which has a $(\lfloor \frac{v-1}{2} \rfloor, \lfloor \frac{v-1}{2} \rfloor)$ -blocking system.*

Proof The trivial (k, k, λ) -BIBD obviously has a $(\lfloor \frac{k-1}{2} \rfloor, \lfloor \frac{k-1}{2} \rfloor)$ -blocking system. Thus, by Lemma 2.3 (a), it can be seen that, for each sufficiently large integer x such that $x(k-1) + 1$ is (k, λ) -admissible, there exists an $(x(k-1) + 1, k, \lambda)$ -BIBD with an $(x \lfloor \frac{k-1}{2} \rfloor, x \lfloor \frac{k-1}{2} \rfloor)$ -blocking system. Since $x \lfloor \frac{k-1}{2} \rfloor \leq \lfloor \frac{x(k-1)}{2} \rfloor$ for all positive integers x , the proof is complete. \square

Note that, for any integer $k \geq 5$, the above lemma implies that a 2-chromatic $(v, k, 1)$ -BIBD exists for each sufficiently large $(k, 1)$ -admissible integer v .

Lemma 3.2 *Let k and λ be positive integers such that $k \geq 5$. For each (k, λ) -admissible integer $m \in \{0, 1, \dots, k(k-1)-1\}$, there is a positive integer z such that $z \geq 2k-1$, $z \equiv m \pmod{k(k-1)}$, and there exists a (z, k, λ) -BIBD which has a $(\lfloor \frac{z-1}{2} \rfloor, \lfloor \frac{z-1}{2} \rfloor)$ -blocking system.*

Proof Let m be a (k, λ) -admissible element of $\{0, 1, \dots, k(k-1)-1\}$. By Lemma 3.1 there exists an integer $y' \equiv 1 \pmod{k(k-1)}$ such that $y' \geq 2k-1$ and there is a (y', k, λ) -BIBD which has a $(\frac{y'-1}{2}, \frac{y'-1}{2})$ -blocking system (note that y' is odd since $k(k-1)$ is even). Thus, since any positive integer congruent to m modulo $k(k-1)$ is itself (k, λ) -admissible, by Lemma 2.4 (a) it can be seen that there is a positive integer x such that $x \equiv m \pmod{k(k-1)}$ and there exists an (xy', k, λ) -BIBD which has an $(\frac{xy'-1}{2}, \frac{xy'-1}{2})$ -blocking system. Since $xy' \geq 2k-1$, $xy' \equiv m \pmod{k(k-1)}$ and $\frac{xy'-1}{2} \leq \lfloor \frac{xy'-1}{2} \rfloor$, the proof is complete. \square

Lemma 3.3 *Let z and λ be positive integers such that $z \in \{6, 7, \dots, 17\}$ and z is $(4, \lambda)$ -admissible. Then there exists a $(z, 4, \lambda)$ -BIBD with a $(\lfloor \frac{z}{2} \rfloor, \lfloor \frac{z}{2} \rfloor)$ -blocking system.*

Proof Let λ_{\min} be the smallest positive integer such that z is $(4, \lambda_{\min})$ -admissible, and note that $\lambda \equiv 0 \pmod{\lambda_{\min}}$. It suffices to find a $(z, 4, \lambda_{\min})$ -BIBD with a $(\lfloor \frac{z}{2} \rfloor, \lfloor \frac{z}{2} \rfloor)$ -blocking system (since we can take $\frac{\lambda}{\lambda_{\min}}$ copies of every block in this design).

For each $z \in \{6, 7, \dots, 17\}$, a $(z, 4, \lambda_{\min})$ -BIBD with a $(\lfloor \frac{z}{2} \rfloor, \lfloor \frac{z}{2} \rfloor)$ -blocking system is given explicitly in [10] or [11]. If z is even this gives us the required result immediately, and in each of the cases $z \in \{7, 9, 11, 13, 15, 17\}$ it is routine to check that the given design in fact admits a $(\lfloor \frac{z}{2} \rfloor, \lfloor \frac{z}{2} \rfloor)$ -blocking system as required. \square

4 Examples of c -chromatic BIBDs

In this section we will construct, for all positive integers c , k and λ with $c \geq 2$, $k \geq 4$ and $(c, k) \neq (2, 4)$, c -chromatic BIBDs with block size k and index λ whose orders satisfy various

congruence conditions (see Lemma 4.4). The reason for these particular congruence conditions will become apparent when we employ these examples in Section 5 to establish the asymptotic existence of c -chromatic BIBDs for each $c \geq 2$. Our approach in this section is inspired by a technique used in [4], and also bears similarities to methods used in [12]. Before proving Lemma 4.4, we require three preliminary lemmas.

Lemma 4.1 *Let c and k be integers such that $k \geq 3$ and $c \geq 2$. Then there exists a c -chromatic partial BIBD with block size k and index 1.*

Proof It was shown in [6] (and later proved constructively in [15]) that for any integers $k' \geq 3$ and $c' \geq 1$ there is a partial BIBD with block size k' and index 1 which has chromatic number at least c' . Let (U, \mathcal{A}) be a partial BIBD with block size k and index 1 which has chromatic number at least c .

Let $\mathcal{A} = \{A_1, A_2, \dots, A_t\}$. We will show that there is an $s \in \{1, 2, \dots, t\}$ such that the partial BIBD $(U, \{A_1, A_2, \dots, A_s\})$ is c -chromatic. We claim that, for each $i \in \{1, 2, \dots, t-1\}$, if $(U, \{A_1, A_2, \dots, A_i\})$ has chromatic number c^\dagger then $(U, \{A_1, A_2, \dots, A_{i+1}\})$ has chromatic number c^\dagger or $c^\dagger + 1$. To see this, observe that we can obtain a $(c^\dagger + 1)$ -colouring of $(U, \{A_1, A_2, \dots, A_{i+1}\})$ by taking a (c^\dagger) -colouring of $(U, \{A_1, A_2, \dots, A_i\})$ and recolouring an arbitrary vertex of A_{i+1} with a colour which is not used in the original colouring. Thus, since $(U, \{A_1\})$ has chromatic number 2 and $(U, \{A_1, A_2, \dots, A_t\})$ has chromatic number at least c , it follows that there is indeed an $s \in \{1, 2, \dots, t\}$ such that $(U, \{A_1, A_2, \dots, A_s\})$ is c -chromatic. \square

Lemma 4.2 *Let k be an integer such that $k \geq 5$ and let p be a prime such that $p \geq k$ and if $k = 5$ then $p \equiv 1 \pmod{4}$. Then there exists a transversal design with group size p and block size k , $(V, \mathcal{G}, \mathcal{B})$, such that*

- (V, \mathcal{B}) has a 2-blocking system such that each set of the blocking system intersects each group in \mathcal{G} in exactly $\frac{p-1}{2}$ points; and
- there is a block $B^* \in \mathcal{B}$ such that $(V, \mathcal{B} \setminus \{B^*\})$ has a 2-blocking system such that each set of the blocking system is disjoint from B^* and intersects each group in \mathcal{G} in exactly $\frac{p-1}{2}$ points.

Proof Let $V = \mathbb{Z}_k \times \mathbb{Z}_p$ and let $\mathcal{G} = \{\{x\} \times \mathbb{Z}_p : x \in \mathbb{Z}_k\}$. Let $I = \{-\frac{p-1}{2}, -\frac{p-3}{2}, \dots, \frac{p-3}{2}, \frac{p-1}{2}\}$. For all $i \in I$ and $j \in \mathbb{Z}_p$ let

$$B_{i,j} = \{(x, ix + j) : x \in \mathbb{Z}_k\}$$

where the second coordinates are considered modulo p (here, we could equally say for all $i \in \mathbb{Z}_p$, but it will help later to consider i as an element of I). Let $\mathcal{B} = \{B_{i,j} : i \in I \text{ and } j \in \mathbb{Z}_p\}$. We claim that $(V, \mathcal{G}, \mathcal{B})$ is a transversal design.

It is easy to see that \mathcal{B} contains exactly p^2 blocks of size k . Also, if a pair of points in different groups appears in the blocks $B_{i,j}$ and $B_{i',j'}$ for some $i, i' \in I$ and $j, j' \in \mathbb{Z}_p$ then it is clear that $i\ell = i'\ell \pmod{p}$ for some $\ell \in \{1, 2, \dots, k-1\}$. So, since $k \leq p$, it follows that $i = i'$ and hence $j = j'$. Thus, $(V, \mathcal{G}, \mathcal{B})$ is indeed a transversal design. We will complete the proof by finding a 2-blocking system for (V, \mathcal{B}) such that each set of the blocking system intersects each group in \mathcal{G} in exactly $\frac{p-1}{2}$ points, and a 2-blocking system for $(V, \mathcal{B} \setminus \{B_{0,0}\})$ such that each set of the blocking system is disjoint from $B_{0,0}$ and intersects each group in \mathcal{G} in exactly $\frac{p-1}{2}$ points.

Let

$$\begin{aligned} S_1 &= (\{0, 1, \dots, k-4\} \times \{1, 2, \dots, \frac{p-1}{2}\}) \cup (\{k-3, k-1\} \times \{0, 1, \dots, \frac{p-3}{2}\}) \cup \\ &\quad (\{k-2\} \times \{\frac{p+1}{2}, \frac{p+3}{2}, \dots, p-1\}) \text{ and} \\ S_2 &= (\{0, 1, \dots, k-4\} \times \{\frac{p+1}{2}, \frac{p+3}{2}, \dots, p-1\}) \cup (\{k-3, k-1\} \times \{\frac{p-1}{2}, \frac{p+1}{2}, \dots, p-2\}) \cup \\ &\quad (\{k-2\} \times \{0, 1, \dots, \frac{p-3}{2}\}). \end{aligned}$$

We claim that $\{S_1, S_2\}$ is a 2-blocking system for (V, \mathcal{B}) . Suppose for a contradiction that there exist $a \in I$ and $b \in \mathbb{Z}_p$ such that $B_{a,b} \cap S_2 = \emptyset$. Then

- (1) $\{ax + b : x \in \{0, 1, \dots, k-4\}\} \subseteq \{0, 1, \dots, \frac{p-1}{2}\};$
- (2) $a(k-3) + b \in \{0, 1, \dots, \frac{p-3}{2}, p-1\};$
- (3) $a(k-2) + b \in \{\frac{p-1}{2}, \frac{p+1}{2}, \dots, p-1\};$ and
- (4) $a(k-1) + b \in \{0, 1, \dots, \frac{p-3}{2}, p-1\}.$

From (1) and (2) it can be seen that $(k-3)|a| \leq \frac{p+1}{2}$, unless $a = \frac{p-1}{2}$, $b = 0$ and $k = 5$ in which case (3) is violated. Thus, if $k \geq 6$ then $|a| \leq \frac{p+1}{6} < \frac{p-1}{4}$ since $p \geq k$, and, if $k = 5$ then $|a| \leq \frac{p-1}{4}$ since $p \equiv 1 \pmod{4}$ in this case. So in all cases $a \in \{-\lfloor \frac{p-1}{4} \rfloor, -\lfloor \frac{p-5}{4} \rfloor, \dots, \lfloor \frac{p-1}{4} \rfloor\}$ and it follows from (1), (2) and (3) that $a(k-1) + b \in \{\frac{p-1}{2}, \frac{p+1}{2}, \dots, p-2\}$, a contradiction to (4). It can be similarly shown that no block in \mathcal{B} is disjoint from S_1 .

Let

$$\begin{aligned} T_1 &= (\{0, 1, \dots, k-3, k-1\} \times \{1, 2, \dots, \frac{p-1}{2}\}) \cup (\{k-2\} \times \{\frac{p+1}{2}, \frac{p+3}{2}, \dots, p-1\}) \text{ and} \\ T_2 &= (\{0, 1, \dots, k-3, k-1\} \times \{\frac{p+1}{2}, \frac{p+3}{2}, \dots, p-1\}) \cup (\{k-2\} \times \{1, 2, \dots, \frac{p-1}{2}\}). \end{aligned}$$

We claim that $\{T_1, T_2\}$ is a 2-blocking system for $(V, \mathcal{B} \setminus \{B_{0,0}\})$. Suppose for a contradiction that there exist $a \in I$ and $b \in \mathbb{Z}_p$ such that $(a, b) \neq (0, 0)$ and $B_{a,b} \cap T_2 = \emptyset$. Then

$$(1) \quad \{ax + b : x \in \{0, 1, \dots, k-3\}\} \subseteq \{0, 1, \dots, \frac{p-1}{2}\};$$

$$(2) \quad a(k-2) + b \in \{0, \frac{p+1}{2}, \frac{p+3}{2}, \dots, p-1\}; \text{ and}$$

$$(3) \quad a(k-1) + b \in \{0, 1, \dots, \frac{p-1}{2}\}.$$

From (1) it can be seen that $(k-3)|a| \leq \frac{p-1}{2}$. Thus, since $k \geq 5$, $|a| \leq \frac{p-1}{4}$. So $a \in \{-\lfloor \frac{p-1}{4} \rfloor, -\lfloor \frac{p-5}{4} \rfloor, \dots, \lfloor \frac{p-1}{4} \rfloor\}$ and it follows from (1) and (2) that $a(k-1)+b \in \{\frac{p+1}{2}, \frac{p+3}{2}, \dots, p-1\}$, a contradiction to (3). It can be similarly shown that no block in $\mathcal{B} \setminus \{B_{0,0}\}$ is disjoint from T_1 . \square

Lemma 4.3 *There exists a transversal design with group size 13 and block size 4, $(V, \mathcal{G}, \mathcal{B})$, such that*

- (V, \mathcal{B}) has a 3-blocking system, each set of which intersects each group in \mathcal{G} in exactly 4 points; and
- there exists a block $B^* \in \mathcal{B}$ such that $(V, \mathcal{B} \setminus \{B^*\})$ has a 3-blocking system each set of which is disjoint from B^* and intersects each group in \mathcal{G} in exactly 4 points.

Also, for each positive integer λ , there exists a $(13, 4, \lambda)$ -BIBD with a $(4, 4, 4)$ -blocking system.

Proof Let $V = \mathbb{Z}_4 \times \mathbb{Z}_{13}$ and $\mathcal{G} = \{\{x\} \times \mathbb{Z}_{13} : x \in \mathbb{Z}_4\}$. For all $i, j \in \mathbb{Z}_{13}$ let

$$B_{i,j} = \{(0, i), (1, j), (2, i+j), (3, i+2j)\},$$

and let $\mathcal{B} = \{B_{i,j} : i, j \in \mathbb{Z}_{13}\}$. Then $(V, \mathcal{G}, \mathcal{B})$ is a transversal design with group size 13 and block size 4. Let

$$S_1 = \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2), (3, 8), (3, 9)\};$$

$$S_2 = \{(0, 5), (0, 6), (0, 7), (0, 8), (1, 5), (1, 6), (1, 7), (1, 10), (2, 7), (2, 8), (2, 9), (2, 12), (3, 0), (3, 3), (3, 10), (3, 11)\};$$

$$S_3 = \{(0, 0), (0, 9), (0, 10), (0, 11), (1, 0), (1, 8), (1, 11), (1, 12), (2, 0), (2, 5), (2, 6), (2, 10), (3, 4), (3, 5), (3, 6), (3, 7)\}.$$

Then $\{S_1, S_2, S_3\}$ is a blocking system for (V, \mathcal{B}) , each set of which intersects each group in \mathcal{G} in exactly 4 points. Let

$$T_1 = \{(0, 2), (0, 3), (0, 4), (0, 5), (1, 2), (1, 3), (1, 4), (1, 5), (2, 1), (2, 3), (2, 4), (2, 5), (3, 1), (3, 2), (3, 10), (3, 11)\};$$

$$T_2 = \{(0, 6), (0, 7), (0, 8), (0, 9), (1, 6), (1, 7), (1, 9), (1, 10), (2, 8), (2, 9), (2, 10), (2, 11), (3, 0), (3, 4), (3, 5), (3, 12)\};$$

$$T_3 = \{(0, 0), (0, 10), (0, 11), (0, 12), (1, 0), (1, 8), (1, 11), (1, 12), (2, 0), (2, 6), (2, 7), (2, 12), (3, 6), (3, 7), (3, 8), (3, 9)\}.$$

Then $\{T_1, T_2, T_3\}$ is a blocking system for $(V, \mathcal{B} \setminus \{B_{1,1}\})$ each set of which is disjoint from $B_{1,1}$ and intersects each group in \mathcal{G} in exactly 4 points.

We saw in the proof of Lemma 3.3 that there exists a $(13, 4, 1)$ -BIBD, and it is easy to show that any such design must have a $(4, 4, 4)$ -blocking system. By taking λ copies of each block in this design, we can obtain the required $(13, 4, \lambda)$ -BIBD. \square

The blocking systems $\{S_1, S_2, S_3\}$ and $\{T_1, T_2, T_3\}$ in the above proof were found by computer search.

Lemma 4.4 *Let c, k, λ and m be positive integers such that $c \geq 2, k \geq 4$ and $m \equiv 0 \pmod{k(k-1)}$. Further suppose that if $k = 4$, then $c \geq 3$ and $m \not\equiv 0 \pmod{13}$. For each (k, λ) -admissible integer $\ell \in \{0, 1, \dots, m-1\}$ there is an integer w such that $w > m, w \equiv \ell \pmod{m}$ and there exists a c -chromatic (w, k, λ) -BIBD with an (s_1, s_2, \dots, s_c) -blocking system for some integers s_1, s_2, \dots, s_c satisfying $s_i \leq \lfloor \frac{w-1}{2} \rfloor$ for each $i \in \{1, 2, \dots, c\}$.*

Proof Let ℓ be a (k, λ) -admissible element of $\{0, 1, \dots, m-1\}$. We will first deal with the case $k \geq 5$. The special case $k = 4$ will be dealt with later.

By Dirichlet's Theorem there are infinitely many primes congruent to 1 modulo $k(k-1)$. Thus, by Lemma 3.1, we can choose p to be an odd prime such that $p \equiv 1 \pmod{k(k-1)}$, $\gcd(p, m) = 1$ and there exists a (p, k, λ) -BIBD with a $(\frac{p-1}{2}, \frac{p-1}{2})$ -blocking system (note that any integer congruent to 1 modulo $k(k-1)$ is (k, λ) -admissible).

By Lemma 4.1 there is a c -chromatic partial BIBD with block size k and index 1. Clearly, by adding points and blocks to this design in such a way that each new block is disjoint from each other block of the design, we can produce, for some positive integer u , a c -chromatic partial $(u, k, 1)$ -BIBD (U, \mathcal{A}_1) such that $\gcd(|\mathcal{A}_1|, m) = 1$ and there is a point in U which is in exactly one block of \mathcal{A}_1 . Let $b = |\mathcal{A}_1|$ and let $\{R_1, R_2, \dots, R_c\}$ be a blocking system for (U, \mathcal{A}_1) .

Let G be the graph on vertex set U in which two vertices are adjacent if and only if the corresponding pair of points is contained in a block in \mathcal{A}_1 . Note that $\gcd(\{\deg_G(x) : x \in V(G)\}) = k-1$ and $|E(G)| = \frac{bk(k-1)}{2}$. Thus, by Theorem 13.1 of [13], there is a decomposition of the λ -fold complete graph of order v into copies of G for all sufficiently large integers v such that $\lambda(v-1) \equiv 0 \pmod{k-1}$ and $\lambda v(v-1) \equiv 0 \pmod{bk(k-1)}$. Since $\gcd(p, m) = 1$ and $\gcd(b, m) = 1$, by the Chinese Remainder Theorem there are infinitely many positive integers which are congruent to 1 modulo b and whose product with p is congruent to ℓ modulo m . Thus, there is an integer

y such that $y \geq 2u$, $py > m$, $py \equiv \ell \pmod{m}$, $y \equiv 1 \pmod{b}$ and there is a decomposition of the λ -fold complete graph of order y into copies of G (note that since ℓ is (k, λ) -admissible, since $p \equiv 1 \pmod{k(k-1)}$, since $m \equiv 0 \pmod{k(k-1)}$, and since $\gcd(b, m) = 1$, the congruences imply that $\lambda(y-1) \equiv 0 \pmod{k-1}$ and $\lambda y(y-1) \equiv 0 \pmod{bk(k-1)}$). Clearly then, there is an embedding of (U, \mathcal{A}_1) in a (y, k, λ) -BIBD $(Y, \mathcal{A}_1 \cup \mathcal{A}_2)$.

Let Z be a set such that $|Z| = p$. Let $z^* \in Z$ and let $\{Z_1, Z_2\}$ be a partition of $Z \setminus \{z^*\}$ such that $|Z_1| = |Z_2| = \frac{p-1}{2}$. Let $V = Y \times Z$ be a point set. Let $S_1 = (Y \times Z_1) \cup (R_1 \times \{z^*\})$, $S_2 = (Y \times Z_2) \cup (R_2 \times \{z^*\})$, and $S_i = R_i \times \{z^*\}$ for each $i \in \{3, 4, \dots, c\}$. Note that S_1, S_2, \dots, S_c are pairwise disjoint and that, since $y \geq 2u$, $|S_i| \leq \lfloor \frac{py-1}{2} \rfloor$ for each $i \in \{1, 2, \dots, c\}$. We will construct a collection of blocks \mathcal{C} such that (V, \mathcal{C}) is a (py, k, λ) -BIBD for which $\{S_1, S_2, \dots, S_c\}$ is a blocking system, and such that \mathcal{C} contains an isomorphic copy of \mathcal{A}_1 . This will complete the proof since $py > m$, since $py \equiv \ell \pmod{m}$ and since the fact that \mathcal{C} contains an isomorphic copy of \mathcal{A}_1 implies that (V, \mathcal{C}) has chromatic number at least c .

For each $A \in \mathcal{A}_1$, let \mathcal{B}_A be a collection of blocks such that $(A \times Z, \{\{x\} \times Z : x \in A\}, \mathcal{B}_A)$ is a transversal design with group size p and block size k such that $A \times \{z^*\} \in \mathcal{B}_A$ and $\{A \times Z_1, A \times Z_2\}$ is a blocking system for $(A \times Z, \mathcal{B}_A \setminus \{A \times \{z^*\}\})$ (such a collection exists by Lemma 4.2, noting that if $k = 5$ then $p \equiv 1 \pmod{4}$ since $p \equiv 1 \pmod{k(k-1)}$). For each $A \in \mathcal{A}_2$, let \mathcal{B}_A^\dagger be a collection of blocks such that $(A \times Z, \{\{x\} \times Z : x \in A\}, \mathcal{B}_A^\dagger)$ is a transversal design with group size p and block size k for which $\{A \times Z_1, A \times Z_2\}$ is a blocking system (such a collection exists by Lemma 4.2, noting that if $k = 5$ then $p \equiv 1 \pmod{4}$ since $p \equiv 1 \pmod{k(k-1)}$). For each $x \in Y$, let \mathcal{B}_x be a collection of blocks such that $(\{x\} \times Z, \mathcal{B}_x)$ is a (p, k, λ) -BIBD for which $\{\{x\} \times Z_1, \{x\} \times Z_2\}$ is a blocking system (such a collection exists by the definition of p).

Let

$$\mathcal{C} = \left(\bigcup_{A \in \mathcal{A}_1} \mathcal{B}_A \right) \cup \left(\bigcup_{A \in \mathcal{A}_2} \mathcal{B}_A^\dagger \right) \cup \left(\bigcup_{x \in Y} \mathcal{B}_x \right).$$

It can be seen that (V, \mathcal{C}) is a (py, k, λ) -BIBD for which $\{S_1, S_2, \dots, S_c\}$ is a blocking system and that \mathcal{C} contains an isomorphic copy of (U, \mathcal{A}_1) (on the point set $U \times \{z^*\}$). This completes the proof in the case $k \geq 5$.

In the case $k = 4$ note that $c \geq 3$ and choose $p = 13$. Note that $p \equiv 1 \pmod{k(k-1)}$, that $\gcd(p, m) = 1$ since $m \not\equiv 0 \pmod{13}$, and that there is a (p, k, λ) -BIBD with a $(4, 4, 4)$ -blocking system by Lemma 4.3. Also by Lemma 4.3, there exists a transversal design with group size 13 and block size 4, $(V, \mathcal{G}, \mathcal{B})$, such that

- (V, \mathcal{B}) has a 3-blocking system, each set of which intersects each group in \mathcal{G} in exactly 4 points; and
- there exists a block $B^* \in \mathcal{B}$ such that $(V, \mathcal{B} \setminus \{B^*\})$ has a 3-blocking system each set of which is disjoint from B^* and intersects each group in \mathcal{G} in exactly 4 points.

By using a similar argument to that used in the case $k \geq 5$ we can obtain the required block design. \square

5 Asymptotic existence of c -chromatic BIBDs

We are now almost ready to prove Theorem 1.1 in the case $k \geq 4$. The final preliminary result we require uses Wilson's fundamental construction to obtain GDDs with a large number of groups of large size which possess 2-blocking systems with certain properties.

Lemma 5.1 *Let k and λ be positive integers such that $k \geq 4$. Then there exist positive integers t and a_0 such that if a , a^\dagger and a^\ddagger are integers such that $a \geq a_0$, $a^\dagger \leq a$, $a^\ddagger \leq a$ and $a \equiv a^\dagger \equiv a^\ddagger \equiv 0 \pmod{k(k-1)}$, then there exists a (k, λ) -GDD of type $a^t(a^\dagger)^1(a^\ddagger)^1$ which has a 2-blocking system such that each set of the blocking system intersects each group G of the GDD in exactly $\frac{|G|}{2}$ points.*

Proof By Lemma 2.2, it is easy to see that there is a positive integer t such that for each $s \in \{t, t+1, t+2\}$ there exists a (k, λ) -GDD of type $(k(k-1))^s$ with a 2-blocking system such that each set of the blocking system intersects each group of the design in exactly $\frac{k(k-1)}{2}$ points. The main result of [2] implies that, for a given positive integer k' , there exists a transversal design with group size g' and block size k' for all sufficiently large integers g' . Thus, there is an integer g_0 such that for any integer $g \geq g_0$ there exists a transversal design with group size g and block size $t+2$. Let a , a^\dagger and a^\ddagger be integers such that $a \geq g_0 k(k-1)$, $a^\dagger \leq a$, $a^\ddagger \leq a$ and $a \equiv a^\dagger \equiv a^\ddagger \equiv 0 \pmod{k(k-1)}$. Then there exists a transversal design with group size $\frac{a}{k(k-1)}$ and block size $t+2$. By deleting some points from this transversal design we can obtain a $(\{t, t+1, t+2\}, 1)$ -GDD $(V, \mathcal{F}, \mathcal{A})$ of type $(\frac{a}{k(k-1)})^t(\frac{a^\dagger}{k(k-1)})^1(\frac{a^\ddagger}{k(k-1)})^1$.

Let Z be a set with $|Z| = k(k-1)$ and let $\{Z_1, Z_2\}$ be a partition of Z with $|Z_1| = |Z_2| = \frac{k(k-1)}{2}$. Let $\mathcal{G} = \{F \times Z : F \in \mathcal{F}\}$. For each block $A \in \mathcal{A}$, let \mathcal{B}_A be a collection of blocks such that

$(A \times Z, \{\{x\} \times Z : x \in A\}, \mathcal{B}_A)$ is a (k, λ) -GDD of type $(k(k-1))^{|A|}$ for which $\{A \times Z_1, A \times Z_2\}$ is a blocking system (such a collection exists since $|A| \in \{t, t+1, t+2\}$ and $|Z_1| = |Z_2| = \frac{k(k-1)}{2}$).

Let

$$\mathcal{B} = \bigcup_{A \in \mathcal{A}} \mathcal{B}_A.$$

It is routine to check that $(V \times Z, \mathcal{G}, \mathcal{B})$ is a (k, λ) -GDD of type $a^t(a^\dagger)^1(a^\ddagger)^1$ for which $\{V \times Z_1, V \times Z_2\}$ is a blocking system. Since $|(V \times Z_1) \cap G| = |(V \times Z_2) \cap G| = \frac{|G|}{2}$ for each $G \in \mathcal{G}$, the proof is complete. \square

Proof of Theorem 1.1 in the case $k \geq 4$. It is known that for any positive integer λ , a 2-chromatic $(v, 4, \lambda)$ -BIBD exists for each $(4, \lambda)$ -admissible integer v (see [7, 10, 11, 17]), so we may assume that if $k = 4$ then $c \geq 3$. Since there are only finitely many congruence classes modulo $k(k-1)$ it suffices to show that, for each (k, λ) -admissible integer $\ell' \in \{0, 1, \dots, k(k-1)-1\}$, there is a c -chromatic (v, k, λ) -BIBD for each sufficiently large integer v such that $v \equiv \ell' \pmod{k(k-1)}$.

Let $\ell' \in \{0, 1, \dots, k(k-1)-1\}$ be a (k, λ) -admissible integer. We will first deal with the case where $k \geq 5$ or where $k = 4$ and ℓ' is odd. The special case where $k = 4$ and ℓ' is even will be dealt with later. By Lemma 3.2 or Lemma 3.3 there is a positive integer u such that $u \equiv \ell' \pmod{k(k-1)}$, if $k \geq 5$ then $u \geq 2k-1$, if $k = 4$ then $u \in \{7, 9, 11, 13, 15, 17\}$, and there exists a (u, k, λ) -BIBD with a $(\lfloor \frac{u-1}{2} \rfloor, \lfloor \frac{u-1}{2} \rfloor)$ -blocking system. Let $m = \text{lcm}(k(k-1), u-1)$. Now, since there are only finitely many congruence classes modulo m it suffices to show that, for each (k, λ) -admissible integer $\ell'' \in \{0, 1, \dots, m-1\}$ for which $\ell'' \equiv \ell' \pmod{k(k-1)}$, there is a c -chromatic (v, k, λ) -BIBD for each sufficiently large integer v such that $v \equiv \ell'' \pmod{m}$.

Let $\ell'' \in \{0, 1, \dots, m-1\}$ be a (k, λ) -admissible integer for which $\ell'' \equiv \ell' \pmod{k(k-1)}$. Note the following facts.

- (i) By Lemma 5.1 there are positive integers t and a_0 such that for any integers x, x^\dagger and x^\ddagger such that $x \geq a_0, x^\dagger \leq x, x^\ddagger \leq x$ and $x \equiv x^\dagger \equiv x^\ddagger \equiv 0 \pmod{k(k-1)}$ there exists a (k, λ) -GDD of type $x^t(x^\dagger)^1(x^\ddagger)^1$ which has a 2-blocking system such that each set of the blocking system intersects each group G of the GDD in exactly $\frac{|G|}{2}$ points.
- (ii) Since there exists a (u, k, λ) -BIBD with a $(\lfloor \frac{u-1}{2} \rfloor, \lfloor \frac{u-1}{2} \rfloor)$ -blocking system, by Lemma 2.3 (b) there is a positive integer n_0 such that, for each integer $n \geq n_0$ with $n \equiv 0 \pmod{m}$, there exists a (k, λ) -GDD of type $u^1 1^n$ which has an $(\frac{n}{2} + \lfloor \frac{u}{2} \rfloor, \frac{n}{2} + \lfloor \frac{u}{2} \rfloor)$ -blocking system such that each set of the blocking system intersects the group of size u in exactly $\lfloor \frac{u}{2} \rfloor$ points (note

that $n + u$ is (k, λ) -admissible since $m \equiv 0 \pmod{k(k-1)}$, that $\lfloor \frac{u-1}{2} \rfloor \leq \lfloor \frac{u}{2} \rfloor$, and that $\frac{n}{u-1} \lfloor \frac{u-1}{2} \rfloor \leq \frac{n}{2}$.

- (iii) By Lemma 4.4 there is an integer $w > m$ such that $w \equiv \ell'' \pmod{m}$ and there exists a c -chromatic (w, k, λ) -BIBD with an (s_1, s_2, \dots, s_c) -blocking system for some integers s_1, s_2, \dots, s_c satisfying $s_i \leq \lfloor \frac{w-1}{2} \rfloor$ for each $i \in \{1, 2, \dots, c\}$ (if $k = 4$ then $u \in \{7, 9, 11, 13, 15, 17\}$ which implies that $m \not\equiv 0 \pmod{13}$).

Let N be the smallest integer such that $N \equiv 0 \pmod{m}$ and $N \geq \max\{a_0, n_0 + m(t-1), w - u\}$. Let $v \geq Nt + n_0 + m(t-1) + w$ be an integer such that $v \equiv \ell'' \pmod{m}$. We will construct a c -chromatic (v, k, λ) -BIBD to complete the proof.

It can be seen that there are unique integers a and a^\dagger such that $a \equiv a^\dagger \equiv 0 \pmod{m}$, $v - w = at + a^\dagger$ and $n_0 \leq a^\dagger \leq n_0 + m(t-1)$ (note that $v \equiv w \pmod{m}$). Now since $v \geq Nt + n_0 + m(t-1) + w$, we have that $at + a^\dagger \geq Nt + n_0 + m(t-1)$ and thus, since $a^\dagger \leq n_0 + m(t-1)$, we have that $a \geq N$. Note that $N \geq a_0$, that $N \geq n_0 + m(t-1) \geq a^\dagger$, that $N \geq w - u$ and, since $m \equiv 0 \pmod{k(k-1)}$, that $a \equiv a^\dagger \equiv w - u \equiv 0 \pmod{k(k-1)}$. Thus, by (i) there exists a (k, λ) -GDD $(V, \mathcal{F} \cup \{F^\dagger, F^\ddagger\}, \mathcal{A})$ of type $a^t(a^\dagger)^1(w-u)^1$, where $|F| = a$ for all $F \in \mathcal{F}$, $|F^\dagger| = a^\dagger$ and $|F^\ddagger| = w - u$, which has a 2-blocking system $\{R_1, R_2\}$ such that $|R_1 \cap F| = |R_2 \cap F| = \frac{|F|}{2}$ for each group $F \in \mathcal{F} \cup \{F^\dagger, F^\ddagger\}$.

Let U be a set disjoint from V such that $|U| = u$ and let U_1 and U_2 be disjoint subsets of U with $|U_1| = |U_2| = \lfloor \frac{u}{2} \rfloor$. We will now construct a c -chromatic (v, k, λ) -BIBD on the point set $V \cup U$.

- For each group $F \in \mathcal{F}$, let \mathcal{B}_F be a collection of blocks such that $(F \cup U, \{U\} \cup \{\{f\} : f \in F\}, \mathcal{B}_F)$ is a (k, λ) -GDD of type $u^1 1^a$ for which $\{(F \cap R_1) \cup U_1, (F \cap R_2) \cup U_2\}$ is a blocking system. Such collections exist by (ii) since $a \geq n_0$, $a \equiv 0 \pmod{m}$, $|F \cap R_1| = |F \cap R_2| = \frac{a}{2}$ for all $F \in \mathcal{F}$, and $|U_1| = |U_2| = \lfloor \frac{u}{2} \rfloor$.
- Let \mathcal{B}^\dagger be a collection of blocks such that $(F^\dagger \cup U, \{U\} \cup \{\{f\} : f \in F^\dagger\}, \mathcal{B}^\dagger)$ is a (k, λ) -GDD of type $u^1 1^{a^\dagger}$ for which $\{(F^\dagger \cap R_1) \cup U_1, (F^\dagger \cap R_2) \cup U_2\}$ is a blocking system. Such a collection exists by (ii) since $a^\dagger \geq n_0$, $a^\dagger \equiv 0 \pmod{m}$, $|F^\dagger \cap R_1| = |F^\dagger \cap R_2| = \frac{a^\dagger}{2}$, and $|U_1| = |U_2| = \lfloor \frac{u}{2} \rfloor$.
- Let \mathcal{B}^\ddagger be a collection of blocks such that $(F^\ddagger \cup U, \mathcal{B}^\ddagger)$ is a c -chromatic (w, k, λ) -BIBD which has a blocking system $\{R_1^\ddagger, R_2^\ddagger, \dots, R_c^\ddagger\}$ such that $R_1^\ddagger \subseteq (F^\ddagger \cap R_1) \cup U_1$, $R_2^\ddagger \subseteq (F^\ddagger \cap R_2) \cup U_2$, and

$\{R_1^\dagger, R_2^\dagger, \dots, R_c^\dagger\}$ is a partition of $F^\dagger \cup U$. Such a collection exists by (iii) since $|(F^\dagger \cap R_1) \cup U_1| = |(F^\dagger \cap R_2) \cup U_2| = \lfloor \frac{w}{2} \rfloor$.

Let $S_1 = (R_1 \setminus F^\dagger) \cup R_1^\dagger$, $S_2 = (R_2 \setminus F^\dagger) \cup R_2^\dagger$, and $S_i = R_i^\dagger$ for each $i \in \{3, 4, \dots, c\}$. Note that $\{S_1, S_2, \dots, S_c\}$ is a partition of $V \cup U$. Let

$$\mathcal{B} = \mathcal{A} \cup \left(\bigcup_{F \in \mathcal{F}} \mathcal{B}_F \right) \cup \mathcal{B}^\dagger \cup \mathcal{B}^\ddagger.$$

It is routine to check that $(V \cup U, \mathcal{B})$ is a c -chromatic (v, k, λ) -BIBD for which $\{S_1, S_2, \dots, S_c\}$ is a blocking system (note that no block in $\mathcal{B} \setminus \mathcal{B}^\dagger$ has more than one point in $F^\dagger \cup U$ and hence no such block can be a subset of any set in $\{S_3, S_4, \dots, S_c\}$).

We now consider the special case where $k = 4$ and ℓ' is even. We proceed exactly as we did in the main case, with four exceptions. Firstly we note that by Lemma 3.3 there is an integer u such that $u \equiv \ell' \pmod{k(k-1)}$, $u \in \{6, 8, 10, 12, 14, 16\}$, and there exists a (u, k, λ) -BIBD with a $(\frac{u}{2}, \frac{u}{2})$ -blocking system. Secondly, we define $m = \text{lcm}(12, u)$. Thirdly, instead of (ii) we instead observe the following.

(ii)' Since there exists a (u, k, λ) -BIBD with a $(\frac{u}{2}, \frac{u}{2})$ -blocking system, by Lemma 2.4 (b) there is a positive integer n_0 such that, for all integers n for which $n \geq n_0$ and $n \equiv 0 \pmod{m}$, there exists a (k, λ) -GDD of type $u^1 1^n$ which has a $(\frac{u+n}{2}, \frac{u+n}{2})$ -blocking system such that each set of the blocking system intersects the group of size u in exactly $\frac{u}{2}$ points (note that $n + u$ is (k, λ) -admissible since $m \equiv 0 \pmod{k(k-1)}$).

Lastly, in our justification of (iii) we must note that $m \not\equiv 0 \pmod{13}$ since $u \in \{6, 8, 10, 12, 14, 16\}$.

Except as noted, the arguments given in the main case hold without any alteration. \square

6 The case of block size 3

It only remains for us to prove Theorem 1.1 in the case $k = 3$. When $\lambda = 1$ this has already been achieved by de Brandes, Phelps and Rödl [4]. In this final section we generalise their result to cover all values of λ . The methods we employ are similar, but not identical, to theirs.

Lemma 6.1 *There exist two transversal designs $(V, \mathcal{G}, \mathcal{B}^\dagger)$ and $(V, \mathcal{G}, \mathcal{B})$ with group size 3 and block size 3 having the same point set and the same group set such that*

- there are two distinct points $x, y \in V$ such that every block which is in \mathcal{B} but not in \mathcal{B}^\dagger contains either x or y ; and
- there is a partition $\{S_1, S_2, S_3\}$ of V such that
 - $|S_i \cap G| = 1$ for all $i \in \{1, 2, 3\}$ and $G \in \mathcal{G}$;
 - $\{S_1, S_2, S_3\}$ is a blocking set for (V, \mathcal{B}^\dagger) ; and
 - $S_1 \in \mathcal{B}$.

Proof Let $V = \mathbb{Z}_3 \times \mathbb{Z}_3$ and let $\mathcal{G} = \{\{x\} \times \mathbb{Z}_3 : x \in \mathbb{Z}_3\}$. Let ρ be the permutation $(0\ 1)$ of \mathbb{Z}_3 . Let $\mathcal{B}^\dagger = \{(0, i), (1, i+j), (2, i+2j+1) : i, j \in \mathbb{Z}_3\}$ and let $\mathcal{B} = \{(0, i), (1, i+j), (2, \rho(i+2j+1)) : i, j \in \mathbb{Z}_3\}$, where the addition is considered modulo 3. It is easy to check that \mathcal{B}^\dagger and \mathcal{B} satisfy the required conditions (take $\{x, y\} = \{(2, 0), (2, 1)\}$ and $S_i = \mathbb{Z}_3 \times \{i - 1\}$ for each $i \in \{1, 2, 3\}$). \square

Lemma 6.2 *Let w and λ be positive integers such that $w \geq 5$ and w is $(3, \lambda)$ -admissible. Then there exists a $(w, 3, \lambda)$ -BIBD with a 3-blocking system such that the sets of the system partition the point set of the BIBD and the sizes of any two sets of the system differ by at most 1.*

Proof In Theorem 18.4 of [3], it is proved that, for each positive integer λ and each $(3, \lambda)$ -admissible integer v , there exists a 3-colourable $(v, 3, \lambda)$ -BIBD. It is easy to verify that in each case the $(v, 3, \lambda)$ -BIBD constructed has a 3-blocking system such that the sets of the system partition the point set of the BIBD and the sizes of any two sets of the system differ by at most 1. \square

Lemma 6.3 *Let $h \in \{0, 1, 2, 3, 4, 5\}$ and let λ be a positive integer such that λ is even if h is even. Then there exists a $(3, \lambda)$ -GDD $(V, \mathcal{G}, \mathcal{B})$ of type $h^1 1^6$ with a 3-blocking system such that the sets of the system partition the point set of the GDD, each set of the system contains at least two points which are in groups of size 1, and the sizes of any two sets of the system differ by at most 1.*

Proof The result follows directly from Lemma 6.2 if $h \in \{0, 1\}$, so assume that $h \in \{2, 3, 4, 5\}$. Let $\lambda_{\min} = 1$ if h is odd and $\lambda_{\min} = 2$ if h is even. It suffices to find a $(3, \lambda_{\min})$ -GDD of type $h^1 1^6$ with a 3-blocking system such that the sets of the system partition the point set of the GDD, each set of the system contains at least two points which are in groups of size 1, and the sizes of any two sets of the system differ by at most 1 (since we can take $\frac{\lambda}{\lambda_{\min}}$ copies of every block in this design).

By the main result of [19] there exists a $(3, \lambda_{\min})$ -GDD $(V, \mathcal{G}, \mathcal{B})$ of type $h^1 1^6$. Let H be a group of the GDD of size h and let $X = V \setminus H$.

If $h \in \{3, 5\}$, then it is easy to see that there is a partition $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ of X such that $\{\{y, x_1, x_2\}, \{y, x_3, x_4\}, \{y, x_5, x_6\}\} \not\subseteq \mathcal{B}$ for each $y \in H$. It is then easy to find a blocking system $\{S_1, S_2, S_3\}$ for (V, \mathcal{B}) such that $\{x_1, x_2\} \subseteq S_1$, $\{x_3, x_4\} \subseteq S_2$, $\{x_5, x_6\} \subseteq S_3$, and the sizes of any two of S_1 , S_2 and S_3 differ by at most 1.

If $h = 2$, then it is easy to see that there is a partition $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ of X such that both blocks in \mathcal{B} which contain $\{x_1, x_2\}$ are disjoint from H and at least one block in \mathcal{B} which contains $\{x_3, x_4\}$ is disjoint from H . It is then easy to find a blocking system $\{S_1, S_2, S_3\}$ for (V, \mathcal{B}) such that $\{x_1, x_2\} \subseteq S_1$, $\{x_3, x_4\} \subseteq S_2$, $\{x_5, x_6\} \subseteq S_3$, and the sizes of any two of S_1 , S_2 and S_3 differ by at most 1.

If $h = 4$, then it is routine to check that there is a partition $\{\{x_1, x_2\}, \{x_3, x_4\}, \{x_5, x_6\}\}$ of X such that at least one block in \mathcal{B} which contains $\{x_1, x_2\}$ is disjoint from H , at least one block in \mathcal{B} which contains $\{x_3, x_4\}$ is disjoint from H , and $\{\{y, x_1, x_2\}, \{y, x_3, x_4\}, \{y, x_5, x_6\}\} \not\subseteq \mathcal{B}$ for each $y \in H$. It is then easy to find a blocking system $\{S_1, S_2, S_3\}$ for (V, \mathcal{B}) such that $\{x_1, x_2\} \subseteq S_1$, $\{x_3, x_4\} \subseteq S_2$, $\{x_5, x_6\} \subseteq S_3$, and the sizes of any two of S_1 , S_2 and S_3 differ by at most 1. \square

The *leave* of a partial BIBD (V, \mathcal{B}) with index 1 is the graph with vertex set V in which a pair of vertices is adjacent if and only if the pair is not contained in any block in \mathcal{B} .

Lemma 6.4 *Let c be a positive integer such that $c \geq 3$. For all sufficiently large even integers v there exists a partial $(v, 3, 1)$ -BIBD which has chromatic number at least c and whose leave is*

- *a perfect matching on v vertices if $v \equiv 0, 2 \pmod{6}$; and*
- *the vertex-disjoint union of a complete graph of order 4 and a perfect matching on $v - 4$ vertices if $v \equiv 4 \pmod{6}$.*

Proof It follows from Lemma 4.1 that for some positive integer u there is a partial $(u, 3, 1)$ -BIBD (U, \mathcal{A}) which has chromatic number c . By the main result of [1], we can embed (U, \mathcal{A}) in a $(u', 3, 1)$ -BIBD for some positive integer u' such that $2u + 1 \leq u' \leq 2u + 5$ and $u' \equiv 1 \pmod{6}$. The main result of [9] then guarantees that, for all even integers v such that $v \geq 4u + 12$, this design can in turn be embedded in a partial $(v, 3, 1)$ -BIBD whose leave is

- a perfect matching on v vertices if $v \equiv 0, 2 \pmod{6}$; and

- the vertex-disjoint union of a copy of $K_{1,3}$ and a perfect matching on $v - 4$ vertices if $v \equiv 4 \pmod{6}$.

Furthermore, checking the construction used to prove the main result of [9] when the design to be embedded has order congruent to 1 modulo 6 and the final design has order congruent to 4 modulo 6 (this appears largely in [8]), we see that if $v \equiv 4 \pmod{6}$, then there will be a block in \mathcal{B} whose removal results in a partial $(v, 3, 1)$ -BIBD whose leave is the vertex-disjoint union of a complete graph of order 4 and a perfect matching on $v - 4$ vertices. \square

Proof of Theorem 1.1 in the case $k = 3$. Let N be the smallest even integer such that for each even integer $u' \geq N$ there exists a partial $(u', 3, 1)$ -BIBD which has chromatic number at least c and whose leave satisfies the conditions of Lemma 6.4.

Let v be a $(3, \lambda)$ -admissible integer such that $v \geq 3N$. We will show that there exists a c -chromatic $(v, 3, \lambda)$ -BIBD. Let u and h be the integers such that $v = 3u + h$, u is even, and $0 \leq h \leq 5$. Then $u \geq N$ and there is a partial $(u, 3, 1)$ -BIBD (U, \mathcal{A}) which has chromatic number at least c and whose leave satisfies the conditions of Lemma 6.4. Let $\mathcal{A} = \{A_1, A_2, \dots, A_t\}$. If $u \equiv 0, 2 \pmod{6}$, then let P^* be a pair of points in U which are adjacent in the leave of (U, \mathcal{A}) . If $u \equiv 4 \pmod{6}$, then let P^* be the set of the four points in U which are mutually adjacent in the leave of (U, \mathcal{A}) . In either case, let \mathcal{P} be a partition of $U \setminus P^*$ into pairs of points such that each pair is adjacent in the leave of (U, \mathcal{A}) .

Let $Z = \{z_1, z_2, z_3\}$ be a set and let H be a set such that $|H| = h$. Let $\{H_1, H_2, H_3\}$ be a partition of H such that any two of $|H_1|$, $|H_2|$ and $|H_3|$ differ by at most 1. Let $V = (U \times Z) \cup H$ be a point set. Let $S_i = (U \times \{z_i\}) \cup H_i$ for each $i \in \{1, 2, 3\}$. We will construct collections of blocks $\mathcal{C}_0, \mathcal{C}_1, \dots, \mathcal{C}_t$ such that

- (i) (V, \mathcal{C}_i) is a $(v, 3, \lambda)$ -BIBD for each $i \in \{0, 1, \dots, t\}$;
- (ii) $\{S_1, S_2, S_3\}$ is a blocking system for (V, \mathcal{C}_0) ;
- (iii) (V, \mathcal{C}_t) contains an isomorphic copy of (U, \mathcal{A}) ;
- (iv) the chromatic number of (V, \mathcal{C}_{i+1}) is at most one more than the chromatic number of (V, \mathcal{C}_i) for each $i \in \{0, 1, \dots, t-1\}$.

From (ii) it will follow that (V, \mathcal{C}_0) has chromatic number at most 3, and from (iii) it will follow that (V, \mathcal{C}_t) has chromatic number at least c . Thus, from (iv) it will follow that (V, \mathcal{C}_j) has chromatic

number c for some $j \in \{0, 1, \dots, t\}$. So it suffices to find such collections of blocks.

For each $A \in \mathcal{A}$, let \mathcal{B}_A^\dagger and \mathcal{B}_A be collections of blocks such that $(A \times Z, \{\{x\} \times Z : x \in A\}, \mathcal{B}_A^\dagger)$ and $(A \times Z, \{\{x\} \times Z : x \in A\}, \mathcal{B}_A)$ are $(3, \lambda)$ -GDDs of type 3^3 such that

- $(A \times \{z_1\}, A \times \{z_2\}, A \times \{z_3\})$ is a blocking system for $(A \times Z, \mathcal{B}_A^\dagger)$;
- $A \times \{z_1\} \in \mathcal{B}_A$; and
- there are two distinct points $x, y \in A \times Z$ such that every block which is in \mathcal{B}_A but not in \mathcal{B}_A^\dagger contains either x or y ;

(such collections exists by Lemma 6.1, taking λ copies of every block of the transversal designs). For each $P \in \mathcal{P}$, let \mathcal{B}_P be a collection of blocks such that $((P \times Z) \cup H, \{H\} \cup \{\{x\} : x \in P \times Z\}, \mathcal{B}_P)$ is a $(3, \lambda)$ -GDD of type h^{116} for which $\{(P \times \{z_1\}) \cup H_1, (P \times \{z_2\}) \cup H_2, (P \times \{z_3\}) \cup H_3\}$ is a blocking system (such a collection exists by Lemma 6.3). Let \mathcal{B}_{P^*} be a collection of blocks such that $((P^* \times Z) \cup H, \mathcal{B}_{P^*})$ is a $(3|P^*| + h, 3, \lambda)$ -BIBD for which $\{(P \times \{z_1\}) \cup H_1, (P \times \{z_2\}) \cup H_2, (P \times \{z_3\}) \cup H_3\}$ is a blocking system (such a collection exists by Lemma 6.2 since v is $(3, \lambda)$ -admissible and $3|P^*| + h \equiv v \pmod{6}$).

For each $k \in \{0, 1, \dots, t\}$, let

$$\mathcal{C}_k = \left(\bigcup_{i=1}^k \mathcal{B}_{A_i} \right) \cup \left(\bigcup_{i=k+1}^t \mathcal{B}_{A_i}^\dagger \right) \cup \left(\bigcup_{P \in \mathcal{P}} \mathcal{B}_P \right) \cup \mathcal{B}_{P^*}.$$

It only remains to show that (i), (ii), (iii) and (iv) hold.

It is routine to check that (i), (ii) and (iii) hold (for (iii), the isomorphic copy of (U, \mathcal{A}) is on the point set $U \times \{z_1\}$). To see that (iv) holds, let $i \in \{0, 1, \dots, t-1\}$ and let c_i be the chromatic number of (V, \mathcal{C}_i) . There are two points x and y of V such that every block which is in $\mathcal{B}_{A_{i+1}}$ but not in $\mathcal{B}_{A_{i+1}}^\dagger$ contains either x or y . This implies that every block which is in \mathcal{C}_{i+1} but not in \mathcal{C}_i contains either x or y . Thus, we can obtain a $(c_i + 1)$ -colouring of (V, \mathcal{C}_{i+1}) by taking a c_i -colouring of (V, \mathcal{C}_i) and recolouring the vertices x and y with a colour which is not used in the original colouring. \square

Acknowledgements

The first author was supported in part by an AARMS postdoctoral fellowship. The second author was supported by research grants from NSERC, CFI and IRIF.

References

- [1] D. Bryant and D. Horsley, A proof of Lindner’s conjecture on embeddings of partial Steiner triple systems, *J. Combin. Des.*, **17** (2009), 63–89.
- [2] S. Chowla, P. Erdős and E.G. Straus, On the maximal number of pairwise orthogonal Latin squares of a given order, *Canad. J. Math.*, **12** (1960), 204–208.
- [3] C.J. Colbourn and A. Rosa, *Triple Systems*, Oxford University Press, Oxford (1999).
- [4] M. de Brandes, K.T. Phelps and V. Rödl, Coloring Steiner triple systems, *SIAM J. Algebraic Discrete Methods*, **3** (1982), 241–249.
- [5] P. Dukes and A.C.H. Ling, Asymptotic existence of resolvable graph designs, *Canad. Math. Bull.*, **50** (2007), 504–518.
- [6] P. Erdős and A. Hajnal, On chromatic number of graphs and set-systems, *Acta Math. Acad. Sci. Hungar.*, **17** (1966), 61–99.
- [7] F. Franek, T.S. Griggs, C.C. Lindner, and A. Rosa, Completing the spectrum of 2-chromatic $S(2, 4, v)$, *Discrete Math.*, **247** (2002), 225–228.
- [8] H.L. Fu, C.C. Lindner and C.A. Rodger, The Doyen-Wilson theorem for minimum coverings with triples, *J. Combin. Des.*, **5** (1997) 341–352.
- [9] H.L. Fu, C.C. Lindner and C.A. Rodger, Two Doyen-Wilson theorems for maximum packings with triples, *Discrete Math.*, **178** (1998), 63–71.
- [10] D.G. Hoffman, C.C. Lindner and K.T. Phelps, Blocking sets in designs with block size 4, *European J. Combin.*, **11** (1990) 451–457.
- [11] D.G. Hoffman, C.C. Lindner and K.T. Phelps, Blocking sets in designs with block size four II, *Discrete Math.*, **89** (1991), 221–229.
- [12] D. Horsley and D.A. Pike, On cycle systems with specified weak chromatic number, *J. Combin. Theory Ser. A* **117** (2010), 1195–1206.
- [13] E.R. Lamken and R.M. Wilson, Decompositions of edge-colored complete graphs, *J. Combin. Theory Ser. A*, **89** (2000), 149–200.
- [14] A.C.H. Ling, On 2-chromatic $(v, 5, 1)$ -designs, *J. Geom.*, **66** (1999), 144–148.
- [15] L. Lovász, On chromatic number of finite set-systems, *Acta Math. Acad. Sci. Hungar.*, **19** (1968), 59–67.
- [16] A. Rosa, Steiner triple systems and their chromatic number, *Acta Fac. Rerum Natur. Univ. Comenian. Math.* **24** (1970), 159–174.
- [17] A. Rosa and C.J. Colbourn, Colorings of block designs, in *Contemporary Design Theory: A Collection of Surveys* (Eds. J.H. Dinitz, D.R. Stinson), John Wiley & Sons, New York (1992), 401–430.
- [18] A. Schrijver, *Theory of Linear and Integer Programming*, John Wiley & Sons, Chichester (1986).
- [19] G. Stern, Tripelsysteme mit Untersystemen, *Arch. Math. (Basel)* **33** (1979/80) 204–208.
- [20] R.M. Wilson, An existence theory for pairwise balanced designs. III. Proof of the existence conjectures, *J. Combinatorial Theory Ser. A* **18** (1975), 71–79.